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Short communication

Virtual age of non-repairable objects

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Abstract

Two definitions of virtual age are considered. The first one is based on the fact that a system's deterioration depends on an environment. In a more severe environment deterioration is more intensive and, therefore, the corresponding virtual age is larger than the calendar age. The second approach defines virtual age at the moment of switching from one regime to another. It is shown that both definitions coincide only for the linear scale transformation in the lifetime distribution functions.

Keywords: Virtual age; Degradation; Failure rate; Mean remaining lifetime; Scale transformation

1. Introduction

Our recent paper [1] was devoted to defining and discussing two types of virtual age: the statistical and the information-based virtual ages. In the current note we investigate further only the first notion, expanding it to the important for the accelerate life testing case. The main goal is to understand the underlying assumptions and to discuss the implications of the considered models. Specifically, we prove that the cumulative exposure model [2], which is widely used in accelerated life testing, is properly justified only for the case of the accelerated life model (ALM) with *a linear scale transformation.*

Two main approaches to defining virtual age will be considered. The first one, which defines "the statistical virtual age" is based on an assumption that lifetimes in different environments are ordered in the sense of the (usual) stochastic ordering. Equivalently, this assumption can be also interpreted in terms of the corresponding ALM. This reasoning also helps in recalculation of age, when one regime (stress) is switched to another. We will show that the defined recalculated virtual age is equal to the statistical virtual age at the moment of switching only for the case of the linear ALM. When this assumption does not hold, additional assumptions should be imposed on the corresponding distribution function after the switching.

2. Statistical virtual age

The content of this section mostly follows Section 3 of Finkelstein [1]. We put more emphasis on justification and interpretation of the model and on a more detailed discussion of the notions involved.

Consider a degrading item which operates in a baseline environment and denote the corresponding Cdf of time to failure by $F_b(t)$. We will use the terms *environment*, *regime* or *stress* interchangeably. By "degrading" we mean that the quality of performance of an item is decreasing in some suitable sense, e.g., a corresponding wear is increasing. We will implicitly assume that degradation or wear is additive, but formally the virtual age can be defined without this assumption as well. Let another statistically identical item be operating in a more severe environment with the Cdf of time to failure $F_s(t)$. Assume for simplicity, that environments are not varying with time and that distributions are absolutely continuous and denoted by $\lambda_b(t)$ and $\lambda_s(t)$ the corresponding failure rates. The time-dependent stresses can be also considered [3]. We want to establish an age correspondence between the systems in two regimes considering the baseline as a reference one. It is reasonable

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to assume that degradation in the second regime is more intensive and therefore the time for accumulating the same amount of degradation or wear is smaller than in the baseline one. Therefore, assume that the corresponding lifetimes are ordered in terms of the (usual) stochastic ordering [4] as

$$\bar{F}_s(t) < \bar{F}_b(t), \quad t \in (0, \infty). \tag{1}$$

It should be emphasized that this is our assumption. Although this ordering naturally models an impact of a more severe environment, other weaker orderings can, in principle, describe probabilistic relationships between the corresponding lifetimes in two regimes (e.g., ordering of the mean values, which, in fact, does not lead to the forthcoming reasoning).

Inequality (1) implies the following equation:

$$F_s(t) = F_b(W(t)), \quad W(0) = 0, \quad t \in (0, \infty), \tag{2}$$

where the function $W(t) > t$ is strictly increasing. This follows after applying the inverse function to both parts of (2):

$$W(t) = F_b^{-1}(F_s(t))$$

and noting that the superposition of two increasing functions is also increasing. Thus, Eq. (2) can be interpreted as a general ALM [3,5] with a scale transformation function $W(t)$. As this function is differentiable, it can be interpreted as an additive degradation function:

$$W(t) = \int_0^t w(u)\,du, \tag{3}$$

where $w(t)$ has a meaning of a degradation rate. Without loosing generality, we assume for convenience that in the baseline environment the degradation rate is equal to 1.

Definition 1. Let t be a chronological age of an item in the baseline environment. Assume that the ALM (2) describes the lifetime of another statistically identical item, which operates in a more severe environment for the same time t.

Then the function $W(t)$ defines the statistical virtual age of the second item in the time scale t of the first one or, equivalently, the inverse function $W^{-1}(t)$ defines the statistical virtual age of the first item in the time scale of the second one.

This definition is, in fact, about age correspondence in different regimes. It can be interpreted in the following way. An item that was operating in a more severe regime for the time t 'acquires' the statistical virtual age $W(t) > t$ which corresponds to the chronological age t, if this item would have been operating in a baseline regime. A similar interpretation holds for the inverse regime sequence. The starting statistical virtual age in this case is an inverse function $W^{-1} < (t)$, which can be easily seen after substituting in Eq. (2) the inverse function $W^{-1}(t)$ instead of t. Usually, we will omit *statistical* in what follows. Therefore, the term *virtual age* will often mean *statistical virtual age* of Definition 1.

When the failure rates (or the corresponding Cdfs) are given, or estimated from the data, the ALM defined by Eq. (2) can be viewed as an equation for obtaining the virtual age $W(t)$:

$$\exp\left\{-\int_0^t \lambda_s(u)\,du\right\} = \exp\left\{-\int_0^{W(t)} \lambda_b(u)\,du\right\}$$
$$\Rightarrow \int_0^t \lambda_s(u)\,du = \int_0^{W(t)} \lambda_b(u)\,du. \tag{4}$$

Hence, the virtual age $W(t)$ is uniquely defined by Eq. (4) and $W(t) \to \infty$, as $t \to \infty$; $W(0) = 0$. Similar to (4), the virtual age $W^{-1}(t)$ is obtained from the following equation:

$$\int_0^t \lambda_b(u)\,du = \int_0^{W^{-1}(t)} \lambda_s(u)\,du.$$

Eq. (4) can be interpreted in terms of the cumulative exposure model [2], i.e., the virtual age $W(t)$ 'produces' the same population cumulative fraction of units failing in a more severe environment as the age t does in the baseline one.

Example 1. Let the failure rates in both regimes be increasing exponential functions, therefore defining the corresponding Gompertz distributions, which are used for describing human mortality:

$$\lambda_b(t) = \alpha \exp\{\beta t\}, \quad \lambda_s(t) = \mu \exp\{\eta t\}, \quad \alpha, \beta, \mu, \eta > 0.$$

Let, for simplicity, $\alpha = \mu = 1$. Then $\eta > \beta$ and the virtual age $W(t)$ is defined by Eq. (4) as

$$W(t) = \frac{\ln[(\eta/\beta)(\exp\{\beta t\} - 1) + 1]}{\eta}.$$

It is clear that, owing to $\eta > \beta$, inequality $W(t) > t$ holds for $t > 0$. A similar result can be obtained for the virtual age $W^{-1}(t)$.

Applying the definition of the failure rate to Eq. (2):

$$\lambda_s(t) = \frac{dF_b(W(t))}{dt}\frac{1}{\bar{F}_b(W(t))} = w(t)\lambda_b(W(t)). \tag{5}$$

If, for instance, the failure rate in a baseline regime is a constant, then $\lambda_s(t)$ is proportional to the rate of degradation $W(t)$.

3. Recalculated virtual age

Let an item start operating in a baseline regime at $t = 0$, which is switched at $t = x$ to a more severe one. The virtual age immediately after the switching, in accordance with Definition 1, is $V_x = W^{-1}(x)$, where the new notation V_x is used for convenience.

Assume now that the governing Cdf after the switching is $F_s(t)$ and that the Cdf of the remaining lifetime is $F_s(t|V_x)$, defined by the following equation:

$$F_s(t|V_x) = 1 - \frac{\bar{F}_s(t + V_x)}{\bar{F}_s(V_x)}. \tag{6}$$

Thus, an item starts operating in the second regime having a starting age V_x defined for the Cdf $F_s(t)$. It should be emphasized that the form of the lifetime Cdf after the switching given by Eq. (6) is *our assumption* and that it does not follow directly from the ALM (2). For example, the starting age could differ from V_x, or (and) the governing distribution could differ from $F_s(t)$ (see also the remarks at the end of this section). On the other hand, we can proceed *on the basis* of ALM (2) in a different, more justified way and obtain the Cdf of an item's lifetime for the whole interval $[0,\infty)$.

According to our interpretation of the previous section, the rate of degradation is 1 in $t\in[0,x)$. Assume that the switching at $t=x$ results in the rate $w(t)>1$ in $[x,\infty)$, where $w(t)$ is defined by the ALM (2)–(3). Note that this is an important assumption on the nature of the impact of regime switching in the framework of the ALM. The alternative option, which is not discussed here, is the corresponding jump from the curve $\lambda_b(t)$ to the curve $\lambda_s(t)$ at $t=x$. This option can be interpreted in the framework of the proportional hazards (PH) model, which is usually not suitable for the lifetime modelling of degrading objects [6]. Under the stated assumptions, the item's lifetime Cdf in $[0,\infty)$ to be denoted by $F_{bs}(t)$ can be written as [3]

$$F_{bs}(t)=\begin{cases}F_b(t), & 0\leqslant t<x,\\ F_b\left(x+\int_x^t w(u)\,du\right), & x\leqslant t<\infty.\end{cases} \quad (7)$$

Transforming the second row on the right-hand side of this equation results in

$$F_b\left(x+\int_x^t w(u)\,du\right)=F_b\left(\int_{\tau(x)}^t w(u)\,du\right)=F_b(W(t)-W(\tau(x))), \quad (8)$$

where $\tau(x)<x$ is uniquely defined from the equation:

$$x=\int_{\tau(x)}^x w(u)\,du=W(x)-W(\tau(x)). \quad (9)$$

Assume firstly, hypothetically that an item is operating in $[0,\infty)$ in the second regime. Eq. (9) means that the corresponding cumulative degradation in $[\tau(x),x)$ in this case is equal to the cumulative degradation in the baseline regime in $[0,x)$, which is x. Therefore, an age of an item in our model (7) just after the switching to a more severe regime can be defined as $\tilde{V}_x=x-\tau(x)$, as if an item was switched into operation at time instant $\tau(x)$. Note that $\tilde{V}_x$ is defined via $\tau(x)$, and that this approach is based on the considered specific model. Let us call $\tilde{V}_x$ the *recalculated virtual age.*

Definition 2. Let an item start operating at $t=0$ in the baseline regime and be switched to a more severe one at $t=x$. Assume that the corresponding Cdf in $[0,\infty)$ is given by Eq. (7).

Then the recalculated virtual age $\tilde{V}_x$ after switching at $t=x$ is defined as $x-\tau(x)$, where $\tau(x)$ is a unique solution of Eq. (9).

We are now interested in comparing V_x with $\tilde{V}_x$ and will show that under certain assumptions these quantities are equal. Eq. (9) has the following solution:

$$\tau(x)=W^{-1}(W(x)-x).$$

As $V_x=W^{-1}(x)$, equation $V_x=\tilde{V}_x$ can be written in the form of the following functional equation:

$$x-W^{-1}(x)=W^{-1}(W(x)-x).$$

Applying operation $W(\cdot)$ to both parts of this equation gives:

$$W(x-W^{-1}(x))=W(x)-x.$$

It is easy to show (see also Example 2) that a linear function $W(t)=wt$ is a solution to this equation. It is also clear that it is the unique solution, as the functional equation $f(x+y)=f(x)+f(y)$ for continuous $f(x)$ has only the linear solution. Therefore, the recalculated virtual age in this case is equal to the statistical virtual age. The following example shows that the function defined by the second row on the right-hand side of Eq. (7) is a segment of the Cdf $F_s(t)$ for $t\geqslant x$ also only for this specific linear case.

Example 2. In accordance with Eqs. (2) and (8):

$$F_b(w(t-\tau(x)))=F_s(t-\tau(x)),$$

where $\tau(x)$ is obtained from a simplified Eq. (8):

$$x=\int_{\tau(x)}^x w\,du\Rightarrow\tau(x)=\frac{x(w-1)}{w}$$

and

$$\tilde{V}_x=x-\tau(x)=x/w;\quad V_x=W^{-1}(x)=x/w.$$

It follows from this example that the Cdf $F_{bs}(t)$ for the linear $W(t)$ can be defined in the way usually referred to in the literature on accelerated life testing (e.g., [2]):

$$F_{bs}(t)=\begin{cases}F_b(t), & 0\leqslant t<x,\\ F_s(t-\tau(x)), & x\leqslant t<\infty.\end{cases}$$

This Cdf can be equivalently written as

$$F_{bs}(t)=\begin{cases}F_b(t), & 0\leqslant t<x,\\ F_s(t-x+\tilde{V}_x), & x\leqslant t<\infty.\end{cases}$$

The Cdf of the remaining at $t=x$ time, in accordance with this equation, is

$$\frac{F_s(t-x+\tilde{V}_x)-F_b(x)}{\bar{F}_b(x)}=F_s(t'|V_x),\quad t-x\equiv t'\geqslant 0,$$

where equations $F_b(x)=F_s(V_x)$ (owing to (2)) and $V_x=\tilde{V}_x$ were used. Note, that the first equation is equivalent to Eq. (4). Therefore, the remaining lifetimes obtained via the rate of degradation concept and via Eq. (6) are equal for the linear scale function: $W(t)=wt$. We think that the first concept is somehow better 'physically motivated'.

The failure rate which corresponds to the Cdf $F_{bs}(t)$ is

$$\lambda_{bs}(x) = \begin{cases} \lambda_b(t), & 0 \leqslant t < x, \\ \lambda_s(t - \tau(x)) = \lambda_s(t - x + V_x), & x \leqslant t < \infty. \end{cases}$$

This form of the failure rate is often referred to as the 'Sedjakin principle' [6]. In his original work, Sedjakin [7] defines a notion of 'resource' as a corresponding cumulative failure rate and assumes that after the switch the further operation of the item depends on the history only via this resource and does not depend on how this resource was 'acquired' previously. This assumption, in fact, leads to Eq. (4), which describes the equality of resources for different regimes, and eventually to the definition of virtual age in our meaning.

What happens if the function $W(t)$ is non-linear? Nothing changes in the first approach, which we recommend to use in this case. The virtual age $V_x = W^{-1}(x)$ is defined in the same way and the Cdf of the remaining lifetime is also defined by Eq. (6). On the other hand, the virtual age $\tilde{V}_x = x - \tau(x)$ can be obtained using the same relations as in the linear case, but now $V_x \neq \tilde{V}_x$ and the second row on the right hand side of (7) cannot be transformed to the segment of the Cdf $F_s(t)$. Therefore, an appealing virtual age interpretation of the age recalculation model at switching (with a governing Cdf $F_s(t)$) does not exist any more. Although we can formally define a different Cdf and the corresponding virtual age as a starting age for $F_s(t)$, this approach does not seem to be as well justified as in the linear case.

4. Concluding remarks

The developed virtual age concept can be also applied to repairable systems. Keeping the notation but not the literal meaning, assume that initially the lifetime of a repairable item is characterized by the Cdf $F_b(t)$ and the imperfect repair changes it to $F_s(t|V_x)$, where V_x is the virtual age just after repair at $t = x$. (As a special case, the distribution can be the same: $F_s(t) = F_b(t)$.) Thus, we have two factors. First, the imperfect repair changes the Cdf from $F_b(t)$ to $F_s(t)$ and it is reasonable to assume that the corresponding lifetimes are ordered as in (1). The parameters of the Cdf $F_b(t)$ could be changed by the repair action. If, e.g., $F_b(t) = 1 - \exp\{-\lambda t^{\alpha}\}$; $\lambda, \alpha > 0$ is a Weibull distribution, then a smaller value of parameter λ will result in ordering (1). Secondly, the remaining after the imperfect repair deterioration defines the corresponding initial (virtual) age V_x for distribution $F_s(t)$, which was called in Finkelstein [8] "the hidden age of the Cdf after the change of parameters". This model describes the dependence between lifetimes before and after the repair, which usually exists for degrading objects. If $V_x = 0$, the lifetimes are independent but the model still can describe some kind of imperfect repair, as ordering (1) holds. Specifically, the consecutive cycles of the geometric process [8] present a relevant example. The g-renewal process of imperfect repair [9] is also another meaningful example.

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